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THEORIES AND ANALYSES OF THIN AND MODERATELY THICK LAMINATED COMPOSITE CURVED BEAMS

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Abstract—A complete and consistent set of equations is derived for the analysis of laminated composite curved beams and closed rings. Equations are developed for thin and moderately thick curved beams. Rotary inertia and shear deformation as well as accurate kinematic relations are included in the moderately thick beam equations. Natural frequencies for simply-supported curved beams are obtained by exact solutions. Extension–bending coupling due to unsymmetric lamination is distinguished from that due to curvature. Effects of shear deformation, rotary inertia, curvature and thickness ratios, and material orthotropy on the natural frequencies are studied.

1. INTRODUCTION

Laminated composite materials have increased considerably in their use during the past two decades. Composite materials are used as mechanical components in various applications (aerospace, automotive, ...). While these components can be beams, plates or shells, the most often encountered among these applications are beams. Beams can be straight or curved. Curved beams can have deep or shallow curvature and can be open or closed. Closed beams are often called rings. This paper addresses laminated curved beams and rings.

Vibration analysis of curved beams was the subject of two survey studies (Markus and Nanasi, 1981; Laura and Maurizi, 1987). These studies list more than 200 references on the subject. Almost all the references dealt with isotropic beams. Only eight of these references dealt with composite sandwich beams. The finite element method was used to study the dynamic response of sandwich curved beams (Ahmed, 1971a, b, 1972). Elements having three, four and five degrees of freedom per node were analysed (Ahmed, 1971a). Shear deformation and rotary inertia effects on the natural frequencies were further studied (Ahmed, 1972). Free and forced vibrations of a three-layer damped ring were investigated (DiTaranto, 1973). The same problem was solved using analytical formulation (Lu, 1976; Nelson, 1977). Transient Response was analysed for three-layer rings (Sagartz, 1977). Damping Properties of Curved Sandwich Beams with Viscoelastic Layer were studied (Tatemichi *et al.*, 1980). Viscoelastic damping in the middle core layer was emphasized. Recently, a consistent set of equations was developed for slightly curved laminated beams (Qatu, 1992) and results were obtained for such beams having different boundary conditions. It was shown that the stretching–bending coupling due to lamination has significant effect on the natural frequencies. The analysis of generally laminated curved beams and closed rings is virtually non-existent.

This paper is concerned with the development of the fundamental equations and energy functionals for laminated composite curved beams and closed rings and presents some results which can be useful for design engineers. Two theories are developed for laminated curved beams. In the first theory, thin beams are studied where effects of shear deformation and rotary inertia are neglected. These effects are considered in the second theory which deals with moderately thick beams. Kinematical relations, force and moment resultants, equations of motion and boundary condition are derived and shown to be consistent for both theories. Stretching–bending coupling due to both curvature and unsymmetrical lamination are distinguished. Natural frequencies are presented for simply-supported open curved beams and closed rings by exact methods. Shear deformation and rotary inertia effects as well as those of the thickness and orthotropy ratios and lamination sequence upon the natural frequencies are studied.

2. FUNDAMENTAL THEORY OF THIN BEAMS

A laminated curved beam is characterized by its middle surface, which is defined by the polar coordinate x (Fig. 1), where

$$x = R\theta. \tag{1}$$

The constant R identifies the radius of curvature of the beam (Fig. 1).

The equations derived for deep shells (Leissa, 1973; Love, 1927) can be specialized to those for curved beams by deleting all derivatives with respect to the out-of-plane axis, and all terms containing displacement in the out-of-plane direction. The Reissner–Naghdi type of shell theory will be used in the specialization because, as will be shown, the equations derived using this theory are consistent. Furthermore, the equations derived here are for curved beams subject to inplane loading and/or vibrating in the inplane.

2.1. *Kinematical relationships*

Middle surface strain and curvature changes are:

$$\varepsilon^{o} = \frac{\partial u_o}{\partial x} + \frac{w_o}{R}, \quad \kappa = -\frac{\partial^2 w_o}{\partial x^2} + \frac{1}{R}\frac{\partial u_o}{\partial x}. \tag{2}$$

The strain at an arbitrary point can be found from

$$\varepsilon = \frac{1}{(1+z)/R}(\varepsilon^{o} + z\kappa), \tag{3}$$

where u_o and w_o are displacements of the beam middle surface in the x and z directions, respectively. The term z/R is small in comparison with unity and can be neglected for thin beams. The well-known Kirchhoff hypothesis normals to the middle surface remain straight and normal, and unstretched in length during deformation (Leissa, 1973; Qatu, 1989) was used. In the following derivation, the subscript o will be dropped from the displacement terms for convenience.

2.2. *Force and moment resultants*

Consider now a laminated curved beam composed of composite material laminae (typically, very thin layers). The fibers of a typical layer may not be parallel to the coordinate in which the beam equations are expressed (i.e. the x-coordinate). The stress–strain equations for an element of material in the kth lamina may be written as (Vinson and Sierakowski, 1986):

$$\sigma = Q_{11}\varepsilon, \tag{4}$$

where σ is the normal stress component; the constant Q_{11} is the elastic stiffness coefficient for the material (Vinson and Sierakowski, 1986).

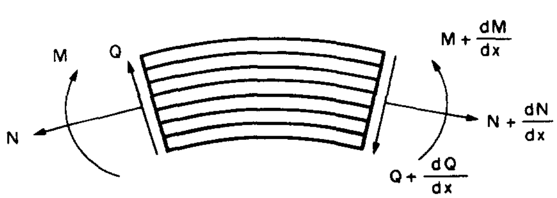

Fig. 1. Laminated curved beam.

The force and moment resultants are the integrals of the stresses over the beam thickness (h):

$$[N, M] = b \int_{-h/2}^{h/2} [1, z]\sigma \, dz, \tag{5}$$

where b is the width of the beam. Keeping in mind that beams of n number of laminates are considered, then the above equations may be rewritten as:

$$[N, M] = b \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} [1, z]\sigma \, dz. \tag{6}$$

Substituting eqn (4) into (6), utilizing eqns (3), after neglecting the z/R term, and carrying out the integration over the thickness piecewise, from layer to layer, yields:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon^{o} \\ \kappa \end{bmatrix}, \tag{7}$$

where A_{11}, B_{11} and D_{11} are the stiffness coefficients arising from the piecewise integration:

$$[A_{11}, B_{11}, D_{11}] = \sum_{k=1}^{N} b\bar{Q}_{11}^{(k)}[(z_k - z_{k-1}), \tfrac{1}{2}(z_k^2 - z_{k-1}^2), \tfrac{1}{3}(z_k^3 - z_{k-1}^3)], \tag{8}$$

where z_k is the distance from the midsurface to the surface of the kth layer having the largest z-coordinate. The above equations are valid for cylindrical bending of plates. For beams with free stress surfaces, the above equations can be modified. This approach can be found in the book by Vinson and Sierakowski (1986).

2.3. *Equations of motion*

The equations of motion may be obtained by taking a differential element of a beam having thickness h and midsurface length dx (Fig. 2), and requiring the sum of the external and internal forces in the x and z direction, and the sum of the external and internal moments in the out-of-plane direction to be zero. The equations of motion therefore become:

$$\frac{\partial N}{\partial x} + \frac{Q}{R} = \bar{\rho}\frac{\partial^2 u}{\partial t^2} - p_x, \tag{9a}$$

$$-\frac{N}{R} + \frac{\partial Q}{\partial x} = \bar{\rho}\frac{\partial^2 w}{\partial t^2} - p_n, \tag{9b}$$

$$\frac{\partial M}{\partial x} - Q = 0, \tag{9c}$$

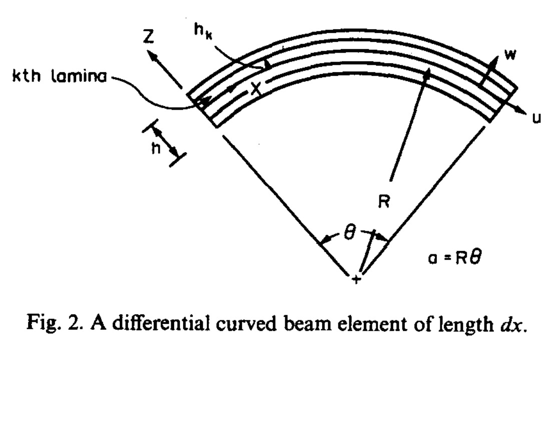

Fig. 2. A differential curved beam element of length dx.

where N, Q and M are the normal and shear forces and the bending moment, respectively, and where p_x and p_n are external force (or body force) components tangent and normal to the beam midsurface, respectively. Solving eqn (9c) for Q, and substituting this into eqn (9b), the equations of motion become:

$$\frac{\partial N}{\partial x}+\frac{1}{R}\frac{\partial M}{\partial x}=\bar{\rho}\frac{\partial^2 u}{\partial t^2}-p_x,$$

$$-\frac{N}{R}+\frac{\partial^2 M}{\partial x^2}=\bar{\rho}\frac{\partial^2 w}{\partial t^2}-p_n. \tag{10}$$

Multiplying the last of eqns (10) through by -1 and substituting eqns (2) and (7) into (10), the equations of motion may be expressed in terms of the midsurface displacements in matrix form as:

$$\begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}\begin{bmatrix} u_o \\ w_o \end{bmatrix}+\begin{bmatrix} -\bar{\rho} & 0 \\ 0 & \bar{\rho} \end{bmatrix}\frac{\partial^2}{\partial t^2}\begin{bmatrix} u_o \\ w_o \end{bmatrix}=\begin{bmatrix} -p_x \\ +p_n \end{bmatrix}, \tag{11}$$

where

$$L_{11}=A_{11}\frac{\partial^2}{\partial x^2}+2\frac{B_{11}}{R}\frac{\partial^2}{\partial x^2}+\frac{D_{11}}{R^2}\frac{\partial^2}{\partial x^2},$$

$$L_{22}=D_{11}\frac{\partial^4}{\partial x^4}-2\frac{B_{11}}{R}\frac{\partial^2}{\partial x^2}+\frac{A_{11}}{R^2},$$

and

$$L_{12}=L_{21}=-B_{11}\frac{\partial^3}{\partial x^3}+\frac{A_{11}}{R}\frac{\partial}{\partial x}+\frac{B_{11}}{R^2}\frac{\partial}{\partial x}-\frac{D_{11}}{R}\frac{\partial^3}{\partial x^3}.$$

For symmetrically laminated beams, all terms containing B_{11} vanish.

2.4. *Energy functionals*

The strain energy stored in a beam during elastic deformation is

$$U=\frac{1}{2}\int_v(\sigma\varepsilon)\,\mathrm{d}V$$

$$=\frac{1}{2}\int(N\varepsilon^o+M\kappa)\,\mathrm{d}x, \tag{12}$$

where V is the volume. Writing the strain energy functional for the kth lamina, and summing for n number of laminates yields

$$U=\frac{1}{2}\int_L[A_{11}(\varepsilon^o)^2+2B_{11}\varepsilon^o\kappa+D_{11}\kappa^2]\,\mathrm{d}x. \tag{13}$$

The above equations can also be derived straightforwardly by substituting eqn (7) into (12). Substituting the strain–displacement and curvature–displacement equations (2) into (13) yields the strain energy functional in terms of the displacements

$$U = \frac{1}{2}\int_L \left(A_{11}\left(\frac{\partial u}{\partial x}+\frac{w}{R}\right)^2 + D_{11}\left(-\frac{\partial^2 w}{\partial x^2}+\frac{1}{R}\frac{\partial u}{\partial x}\right)^2 + 2B_{11}\left(\frac{\partial u}{\partial x}+\frac{w}{R}\right)\left(-\frac{\partial^2 w}{\partial x^2}+\frac{1}{R}\frac{\partial u}{\partial x}\right)\right)\mathrm{d}x. \quad (14)$$

Using the distributed external force components p_x in the tangential (polar) direction, and p_n in the normal direction, the work done by the external forces as the beam displaces is

$$W = \int_L (p_x u + p_n w)\,\mathrm{d}x. \quad (15)$$

The kinetic energy for each lamina is

$$T_k = \tfrac{1}{2} b\rho^{(k)} \int_{L_k}\int_{z_{k-1}}^{z_k} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] \mathrm{d}z\,\mathrm{d}x_k, \quad (16)$$

where $\rho^{(k)}$ is the lamina density per unit volume, and t is time. The kinetic energy of the entire beam is (neglecting rotary inertial terms):

$$T = \sum_{k=1}^{N} T_k = \frac{\bar{\rho}}{2}\int \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x, \quad (17)$$

where $\bar{\rho}$ is the average mass density of the beam per unit length.

The above energy expressions can be shown to be consistent with the equations of motion, strain–displacement relations and boundary conditions by means of Hamilton's Principle. This requires taking the variation of the energy functional $(U-T-W)$, at time t,

$$\delta(U-T-W) = \int \left(N\delta\varepsilon^{o} + M\delta\kappa - \bar{\rho}\frac{\partial u}{\partial t}\delta\frac{\partial u}{\partial t} - \bar{\rho}\frac{\partial w}{\partial t}\delta\frac{\partial w}{\partial t} - p_x\delta u - p_n\delta w \right)\mathrm{d}x$$

$$= \int \left(-\frac{\partial N}{\partial x}\delta u + \frac{N}{R}\delta w - \frac{\partial^2 M}{\partial x^2}\delta w - \frac{1}{R}\frac{\partial M}{\partial x}\delta u - p_x\delta u - p_n\delta w + \bar{\rho}\frac{\partial^2 u}{\partial t^2}\delta u + \bar{\rho}\frac{\partial^2 w}{\partial t^2}\delta w \right)\mathrm{d}x$$

$$+ \left(\left(N + \frac{M}{R}\right)\delta u \Big|_{-a/2}^{a/2} + \frac{\partial M}{\partial x}\delta w \Big|_{-a/2}^{a/2} - M\delta\left(\frac{\partial w}{\partial x}\right)\Big|_{-a/2}^{a/2} \right)$$

$$+ \text{initial conditions.} \quad (18)$$

Hamilton's principle requires the coefficients of δu and δw to vanish independently. The two equations of motion (10) can be obtained by separating the terms multiplied by the variation of u (i.e. δu), and w, respectively.

2.5. *Boundary conditions*

Boundary conditions can be obtained from the last terms of the variational equation. On each boundary one must specify three conditions:

$$u = 0 \quad \text{or} \quad N + \frac{M}{R} = 0, \tag{19a}$$

$$w = 0 \quad \text{or} \quad Q = 0, \tag{19b}$$

$$\frac{\partial w}{\partial x} = 0 \quad \text{or} \quad M = 0. \tag{19c}$$

Note that there is an additional term (M/R) in the boundary condition (19a). This term does not exist for straight or slightly curved beams (Qatu, 1992). Boundaries may also be elastically constrained, with the constraints being represented as translational and rotational springs at the beam edges. In such cases the boundary conditions are generalized to

$$k_u u + \left(N + \frac{M}{R}\right) = 0,$$
$$k_w w + Q = 0,$$
$$k_\psi \frac{\partial w}{\partial x} - M = 0, \tag{20}$$

at the edge $x = -a/2$, where k_u and k_w are translational spring stiffnesses and k_ψ is the rotational spring stiffness. The signs of N, M and Q in the above equations change at the edge $x = a/2$.

A complete and consistent set of equations has been developed for the static and dynamic analysis of laminated composite curved thin beams. By introducing the stiffness and kinematical relationships into the equations of motion, they have been expressed in terms of displacements. If the assumed displacement functions have sufficient continuity, then the equations of compatibility are identically satisfied.

3. FUNDAMENTAL THEORY OF MODERATELY THICK BEAMS

The equations derived for moderately thick beams are different from those derived for thin beams. Shear deformation and rotary inertia are included in the derivation. An exact formulation is made for the stress resultants of thick beams including the z/R term in eqn (3).

3.1. *Kinematical relationships*

Middle surface strain and curvature change are:

$$\varepsilon^o = \frac{\partial u_o}{\partial x} + \frac{w_o}{R}, \quad \kappa = \frac{\partial \psi}{\partial x}, \tag{21}$$

where

$$\gamma = \frac{\partial w}{\partial x} + \psi - \frac{u}{R} \tag{22}$$

and where γ is the shear strain at the neutral axis and ψ is rotation of a line element originally perpendicular to the longitudinal direction, about the out-of-plane direction.

Normal strain at an arbitrary point can be found from

$$\varepsilon = \frac{1}{(1+z/R)}(\varepsilon^{\circ}+z\kappa). \tag{23}$$

In the following derivation, the subscript o will be dropped from the displacement terms for convenience.

3.2. *Force and moment resultants*

The force and moment resultants are the integrals of the stresses over the beam thickness (h):

$$[N, M, Q] = b\int_{-h/2}^{h/2}[\sigma, \sigma z, \tau]\,\mathrm{d}z, \tag{24}$$

where b is the width of the beam. The above equations may be rewritten as:

$$[N, M, Q] = b\sum_{k=1}^{n}\int_{h_{k-1}}^{h_k}[\sigma, \sigma z, \tau]\,\mathrm{d}z. \tag{25}$$

Substituting eqn (21) into (25), utilizing eqns (23), and carrying out the integration over the thickness yields:

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} & 0 \\ B_{11} & D_{11} & 0 \\ 0 & 0 & A_{55} \end{bmatrix}\begin{bmatrix} \varepsilon^{\circ} \\ \kappa \\ \gamma \end{bmatrix}, \tag{26}$$

where the A_{11}, B_{11}, D_{11} and A_{55} are the stiffness coefficients arising from the integration (the z/R term of eqn (23) is included)

$$A_{11} = R\sum_{k=1}^{N} b\bar{Q}_{11}^{(k)}\ln\left(\frac{R+h_k}{R+h_{k-1}}\right),$$

$$B_{11} = R\sum_{k=1}^{N} b\bar{Q}_{11}^{(k)}\left((h_k-h_{k-1})-R\ln\left(\frac{R+h_k}{R+h_{k-1}}\right)\right),$$

$$D_{11} = R\sum_{k=1}^{N} b\bar{Q}_{11}^{(k)}\left(\tfrac{1}{2}([R+h_k]^2-[R+h_{k-1}]^2)-2R(h_k-h_{k-1})+R^2\ln\left(\frac{R+h_k}{R+h_{k-1}}\right)\right),$$

$$A_{55} = \frac{5}{4}\sum_{k=1}^{N} b\bar{Q}_{55}^{(k)}\left[(h_k-h_{k-1})-\frac{4}{3h^2}(h_k^3-h_{k-1}^3)\right]. \tag{27}$$

Table 1 shows a comparison between the above terms when both eqns (27) and (8) are used. Equations (27) are accurate and the comparison should give a hint on the limitation of the constitutive relations used in the thin beam theory. The difference becomes large as the orthotropy ratio increases. This difference reaches approximately 8% for beams with orthotropy ratios of 15 and 40, and thickness ratio h/R of 0.2, which is taken as the limit of thick beam theories. This should also be a benchmark for establishing the limitations of shell equations. In general, the z/R term should be included to obtain accurate equations for thick beams (or shells).

Table 1. A comparison between approximate and accurate equations for the coefficients A_{11}, B_{11} and D_{11} of [0, 90] curved beams

h/r	Approximate equations (8) A_{11}/E_2bR	B_{11}/E_2bR^2	D_{11}/E_2bR^3	Accurate equations (27) A_{11}/E_2bR	B_{11}/E_2bR^2	D_{11}/E_2bR^3
$E_1/E_2 = 1$ (i.e. single layer)						
0.01	0.01	0	0.00000008	0.01000008	−0.00000008	0.00000008
0.02	0.02	0	0.00000067	0.02000067	−0.00000067	0.00000067
0.05	0.05	0	0.00001042	0.05001004	−0.00001042	0.00001042
0.10	0.10	0	0.00008333	0.10008346	−0.00008346	0.00008346
0.20	0.20	0	0.00066667	0.20067070	−0.00067070	0.00067070
Δ†	−0.3%	—	−0.6%			
$E_1/E_2 = 15$						
0.01	0.08	0.000175	0.00000067	0.07982566	0.00017434	0.00000066
0.02	0.16	0.000700	0.00000533	0.15930530	0.00069470	0.00000530
0.05	0.40	0.004375	0.00008333	0.39570700	0.00429300	0.00008200
0.10	0.80	0.017500	0.00066667	0.78314576	0.01685424	0.00064576
0.20	1.60	0.070000	0.00533333	1.53501321	0.06498679	0.00501321
Δ†	4.2%	7.7%	6.4%			
$E_1/E_2 = 40$						
0.01	0.205	0.0004875	0.00000171	0.20451420	0.00048579	0.00000170
0.02	0.410	0.0019500	0.00001367	0.40806357	0.00193643	0.00001357
0.05	1.025	0.0121875	0.00021354	1.01302231	0.01197769	0.00020981
0.10	2.050	0.0487500	0.00170833	2.00289986	0.04710014	0.00164986
0.20	4.100	0.1950000	0.01366667	3.91776771	0.18223229	0.01276771
Δ†	4.7%	7.0%	7.0%			

† The percentage difference between eqns (8) and (27) for $h/R = 0.2$.

3.3. *Equations of motion*

Considering rotary inertia terms, the equations of motion become:

$$\frac{\partial N}{\partial x}+\frac{Q}{R}=\bar{\rho}\frac{\partial^2 u}{\partial t^2}+\bar{Q}\frac{\partial^2 \psi}{\partial t^2}-p_x, \tag{28a}$$

$$-\frac{N}{R}+\frac{\partial Q}{\partial x}=I_1\frac{\partial^2 w}{\partial t^2}-p_n, \tag{28b}$$

$$\frac{\partial M}{\partial x}-Q=\bar{Q}\frac{\partial^2 u}{\partial t^2}+I\frac{\partial^2 \psi}{\partial t^2}, \tag{28c}$$

where $\bar{\rho}$, $\bar{Q}$ and I are defined by

$$(\bar{\rho},\bar{Q},I)=\left(I_1+\frac{I_2}{R}+\frac{I_3}{R^2},I_2+\frac{I_3}{R},I_3\right) \tag{29a}$$

and where

$$[I_1,I_2,I_3]=\sum_{k=1}^{N} b\rho^{(k)}((h_k-h_{k-1}),\tfrac{1}{2}(h_k^2-h_{k-1}^2),\tfrac{1}{3}(h_k^3-h_{k-1}^3)). \tag{29b}$$

It will be shown that the above equations are consistent with the energy functionals. Multiplying the second equation (28b) of the equations of motion through by −1 and substituting eqns (22) and (26) into (28), the equations of motion may be expressed in terms of the midsurface displacements and slope in matrix form as:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u \\ w \\ \psi \end{bmatrix} + \begin{bmatrix} \bar{\rho} & 0 & \bar{Q} \\ 0 & -I_1 & 0 \\ \bar{Q} & 0 & I \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u \\ w \\ \psi \end{bmatrix} = \begin{bmatrix} -p_x \\ +p_n \\ 0 \end{bmatrix}, \tag{30}$$

where

$$L_{11} = A_{11}\frac{\partial^2}{\partial x^2} - \frac{A_{55}}{R^2},$$

$$L_{12} = L_{21} = \left(\frac{A_{11}}{R} + \frac{A_{55}}{R}\right)\frac{\partial}{\partial x},$$

$$L_{13} = L_{31} = B_{11}\frac{\partial^2}{\partial x^2} + \frac{A_{55}}{R^2},$$

$$L_{22} = \frac{A_{11}}{R^2} - A_{55}\frac{\partial^2}{\partial x^2},$$

$$L_{23} = L_{32} = B_{11}\frac{\partial}{\partial x} - A_{55}\frac{\partial}{\partial x},$$

$$L_{33} = D_{11}\frac{\partial^2}{\partial x^2} + A_{55}.$$

For symmetrically laminated beams, all terms containing B_{11} vanish.

3.4. *Energy functionals*

The strain energy stored in a beam during elastic deformation is

$$U = \frac{1}{2}\int\left(N\varepsilon^o + M\frac{\partial\psi}{\partial x} + Q\gamma\right)\mathrm{d}x. \tag{31}$$

Writing the strain energy functional for the kth lamina, and summing for n number of laminates yields

$$U = \frac{1}{2}\int_L [A_{11}(\varepsilon^o)^2 + 2B_{11}\varepsilon^o\kappa + D_{11}\kappa^2 + Q\gamma^2]\,\mathrm{d}x. \tag{32}$$

The above equations can also be derived straightforwardly by substituting eqn (26) into (31). Substituting (21) and (22) into the above energy expression yields the strain energy functional in terms of the slope and displacements

$$U = \frac{1}{2}\int_L \left(A_{11}\left(\frac{\partial u}{\partial x} + \frac{w}{R}\right)^2 + D_{11}\left(\frac{\partial\psi}{\partial x}\right)^2 + 2B_{11}\left(\frac{\partial u}{\partial x} + \frac{w}{R}\right)\frac{\partial\psi}{\partial x} + A_{55}\left(\frac{\partial w}{\partial x} + \psi - \frac{u}{R}\right)^2\right)\mathrm{d}x. \tag{33}$$

The work done by the external force components (p_x in the tangential (polar) direction, and p_n in the normal direction) as the beam displaces is

$$W = \int_L (p_x u + p_n w)\,\mathrm{d}x. \tag{34}$$

The kinetic energy for the entire beam (including rotary inertia) is

$$T = \frac{b}{2}\int_{L_k}\int \rho\left[\left(\left(1+\frac{z}{R}\right)\frac{\partial u}{\partial t} + z\frac{\partial \psi}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}z\,\mathrm{d}x$$

$$= \frac{b}{2}\int_{L}\int \rho\left[\left(\left(1+\frac{z}{R}\right)\frac{\partial u}{\partial t}\right)^2 + 2\left(1+\frac{z}{R}\right)z\frac{\partial u}{\partial t}\frac{\partial \psi}{\partial t}\right.$$

$$\left. + \left(z\frac{\partial \psi}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}z\,\mathrm{d}x. \tag{35a}$$

Integrating over z and summing for N number of layers yields

$$T = \sum_{k=1}^{N} T_k = \frac{b}{2}\int_L \left[\bar{\rho}\left(\frac{\partial u}{\partial t}\right)^2 + I_1\left(\frac{\partial w}{\partial t}\right)^2 + 2\bar{Q}\left(\frac{\partial \psi}{\partial t}\frac{\partial u}{\partial t}\right) + I\left(\frac{\partial \psi}{\partial t}\right)^2\right]\mathrm{d}x. \tag{35b}$$

Hamilton's principle is used again to show that the above energy functionals are consistent with the equations of motion and the kinematical relations. This requires taking the variation of the energy functional $(U-T-W)$, at time t,

$$\delta(U-T-W) = \int\left(N\delta\varepsilon^{\circ} + M\delta\kappa + Q\delta\gamma - \bar{\rho}\delta\frac{\partial u}{\partial t}\frac{\partial u}{\partial t} - I_1\delta\frac{\partial w}{\partial t}\frac{\partial w}{\partial t}\right.$$

$$\left. - \bar{Q}\frac{\partial \psi}{\partial t}\delta\frac{\partial u}{\partial t} - \bar{Q}\frac{\partial u}{\partial t}\delta\frac{\partial \psi}{\partial t} - I\frac{\partial \psi}{\partial t}\delta\frac{\partial \psi}{\partial t} - p_x\delta u - p_n\delta w\right)\mathrm{d}x$$

$$= \int\left(-\frac{\partial N}{\partial x}\delta u + \frac{N}{R}\delta w - \frac{\partial M}{\partial x}\delta\psi - \frac{\partial Q}{\partial x}\delta w + Q\delta\psi - \frac{Q}{R}\delta u - p_x\delta u - p_n\delta w\right.$$

$$\left. + \bar{\rho}\frac{\partial^2 u}{\partial t^2}\delta u + I_1\frac{\partial^2 w}{\partial t^2}\delta w + \bar{Q}\frac{\partial^2 \psi}{\partial t^2}\delta u + \bar{Q}\frac{\partial^2 u}{\partial t^2}\delta\psi + I\frac{\partial^2 \psi}{\partial t^2}\delta\psi\right)\mathrm{d}x$$

$$+ \left(N\delta u\Big|_{-a/2}^{a/2} + Q\delta w\Big|_{-a/2}^{a/2} - M\delta\psi\Big|_{-a/2}^{a/2}\right)$$

$$+\text{initial conditions}. \tag{36}$$

Again, the equations of motion can be obtained by separating the terms multiplied by the variations of u (i.e. δu), w and ψ, respectively.

3.5. *Boundary conditions*

One can get the consistent boundary conditions from the last terms of the variational equation. On each boundary one must specify three conditions:

$$u = 0 \quad \text{or} \quad N = 0, \tag{37a}$$

$$w = 0 \quad \text{or} \quad Q = 0, \tag{37b}$$

$$\psi = 0 \quad \text{or} \quad M = 0. \tag{37c}$$

It is worth mentioning that for moderately thick curved beams, the term M/R in (37a) drops. This is hard to see without the variational derivation. For elastically constrained boundaries, with the constraints being represented as translational and rotational springs at the beam edges, the boundary conditions are generalized to

$$
\begin{aligned}
k_u u + N &= 0, \\
k_w w + Q &= 0, \\
k_\psi \psi - M &= 0
\end{aligned}
\tag{38}
$$

at the edge $x = -a/2$. The signs of N, M and Q in the above equations change at the edge $x = a/2$.

This completes the set of equations for the analysis of laminated composite moderately thick curved beams. The equations of compatibility will be identically satisfied if the assumed displacement functions have sufficient continuity, which is less than that of thin beams.

4. SIMPLY-SUPPORTED BEAMS AND CLOSED RINGS

4.1. *Thin beams*

The simple-support boundary conditions can take two forms for curved beams, namely

$$
\begin{aligned}
&S1: w = N_x = M_x = 0 \quad \text{on} \quad x = -a/2, a/2, \\
&S2: w = u = M_x = 0 \quad \text{on} \quad x = -a/2, a/2.
\end{aligned}
\tag{39}
$$

The above S1 boundary conditions are exactly satisfied by choosing:

$$
[u, w] \sum_{m=1}^{M} A_m [\bar{u}_m, \bar{w}_m] \sin(\omega t), \tag{40}
$$

where $\bar{u}_m = \sin(\alpha x)$, $\bar{w}_m = \cos(\alpha x)$ and $\alpha = m\pi/a$.

The external forces can be expanded in a Fourier series in x:

$$
[p_x, p_z] = \sum_{m=1}^{M} [p_{xm}\bar{u}_m, p_{zm}\bar{w}_m] \sin(\omega t), \tag{41}
$$

where

$$
p_{xm} = \frac{2}{a} \int p_x \bar{u}_m \, dx \quad \text{and} \quad p_{zm} = \frac{2}{a} \int p_z \bar{w}_m \, dx.
$$

Substituting eqns (40) and (41) into the equations of motion written in terms of displacement (11) yields

$$
\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \omega^2 \begin{bmatrix} \bar{\rho} & 0 \\ 0 & -\bar{\rho} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{42}
$$

where

$$
\begin{aligned}
C_{11} &= -\alpha^2 [A_{11} + 2B_{11}/R + D_{11}/R^2], \\
C_{22} &= +(D_{11}\alpha^4) + 2(B_{11}/R)\alpha^2 + (A_{11}/R^2)
\end{aligned}
$$

and

$$C_{21} = -C_{12} = [B_{11}\alpha^3]+\alpha[(A_{11}/R)+(B_{11}/R^2)]+D_{11}/R.$$

Equations (42) are actually valid for problems of forced vibrations with forcing frequency ω. The static problem results when $\omega = 0$. The free vibration problem arises by setting the pressure terms equal to zero.

If one assumes that $\bar{u}_m = \cos(\alpha x)$ and $\bar{w}_m = \sin(\alpha x)$ in eqn (40) then the problem of laminated curved beams having vertical hinge supports (i.e. $u = (\partial w/\partial x) = Q = 0$) is solved. This will yield a frequency determinant which is the same as that of simply-supported beams, and consequently the same natural frequencies.

4.2. *Moderately thick beams*

Similar to thin curved beams, the simple-support boundary conditions for thick beams can take two forms, namely

$$\begin{aligned} &\text{S1}: w = N_x = \frac{\partial\psi}{\partial x} = 0 \quad \text{on} \quad x = -a/2, a/2, \\ &\text{S2}: w = u = \frac{\partial\psi}{\partial x} = 0 \quad \text{on} \quad x = -a/2, a/2. \end{aligned} \tag{43}$$

The above S1 boundary conditions are exactly satisfied by choosing:

$$[u, w, \psi] = \sum_{m=1}^{M} [A_m\bar{u}_m, C_m\bar{w}_m, B_m\bar{\psi}_m]\sin(\omega t), \tag{44}$$

where $\bar{u}_m = \sin(\alpha x)$, $\bar{w}_m = \cos(\alpha x)$, $\bar{\psi}_m = \sin(\alpha x)$ and $\alpha = m\pi/a$.

The external forces can be expanded in a Fourier series as in eqn (41). Substituting eqns (43) and (44) into the equations of motion written in terms of displacement for moderately thick beams (30) yields

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \omega^2 \begin{bmatrix} \bar{\rho} & 0 & \bar{Q} \\ 0 & -I_1 & 0 \\ \bar{Q} & 0 & I \end{bmatrix} \begin{bmatrix} A_m \\ C_m \\ B_m \end{bmatrix} + \begin{bmatrix} p_{xm} \\ -p_{zm} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{45}$$

where

$$C_{11} = -\alpha^2 A_{11} - A_{55}/R^2,$$

$$C_{22} = A_{11}/R^2 + A_{55}\alpha^2,$$

$$C_{33} = -D_{11}\alpha^2 - A_{55},$$

$$C_{21} = -C_{12} = \alpha[(A_{11}/R)+(A_{55}/R)],$$

$$C_{31} = C_{13} = -B_{11}\alpha^2 + (A_{55}/R)$$

and

$$C_{23} = -C_{32} = \alpha[(B_{11}/R) - A_{55}].$$

Again, eqns (45) are actually valid for problems of forced vibrations with forcing frequency ω. The static problem results when $\omega = 0$ and the free vibration problem arises when the pressure terms are zero.

If one assumes that $\bar{u}_m = \cos(\alpha x)$, $\bar{w}_m = \sin(\alpha x)$ and $\bar{\psi}_m = \cos(\alpha x)$ in eqn (44) then the problem of a laminated curved beam having vertical hinge supports (i.e. $u = \psi = Q = 0$) is solved.

It should be mentioned that the above exact solutions are exact with respect to the theory presented here. They are not exact with respect to the theory of elasticity.

4.3. *Closed rings*

The analysis of closed rings can be performed by assuming $\bar{u}_m = \sin(m\theta)$, $\bar{w}_m = \cos(m\theta)$, in eqns (40) and integrating over θ instead of x using eqn (1). This will yield a determinant similar to that of (42) for thin rings with replacing α by m/R. Similar treatment can be made for moderately thick rings by further assuming $\bar{\psi}_m = \sin(m\theta)$, which will yield a third order determinant similar to that of (45).

5. NUMERICAL RESULTS

Table 2 shows a comparison of the first four natural frequencies obtained for simply-supported beams using equations of slightly curved beams (Qatu, 1992) and those presented here for thin beams with deep curvature. The non-dimensional frequency parameter $\Omega = \omega a^2\sqrt{12\rho/E_1h^2}$ [$= \omega a^2\sqrt{\rho A/E_1 I}$, where A is the cross-sectional area and I is the moment of inertia $bh^3/12$ (Leissa, 1991)] is used in the comparison. As shown in the results, the shallow beam equations give higher frequencies than the equations of deep beams. As the curvature increases, the difference between both theories increases and reaches a maximum of approximately 10% for the fundamental frequency when a/R reaches 1, which is taken as the limit of the shallow beam equations. This difference decreases for high frequencies because the mode shapes associated with the higher frequencies divide the beam into sections, each of which is shallower than the original beams.

Table 3 shows a comparison between the results obtained using thin and moderately thick curved beam equations. The same frequency parameter is used. The thickness ratio h/R is varied from 0.01 to 0.2, which is taken as the limit of the moderately thick beam equations. The results show a criterion for establishing the limit of the thin beam theory.

Table 2. Exact frequency parameters Ω for simply-supported [0, 90] laminated curved thin beams using deep and shallow beam theories, $a/h = 100$

	Deep beam theory [eqn (42)]				Shallow beam theory (Qatu, 1992)			
	m				m			
a/R	1	2	3	4	1	2	3	4
$E_1/E_2 = 1$ (i.e. single layer)								
0.0	9.8702	39.491	88.892	158.12	9.8702	39.477	88.826	157.91
0.1	9.8549	39.475	88.876	158.11	9.8642	39.473	88.820	157.91
0.2	9.8102	39.431	88.830	158.06	9.8493	39.460	88.807	157.89
0.3	9.7364	39.356	88.757	157.99	9.8249	39.434	88.781	157.87
0.5	9.4993	39.116	88.516	157.75	9.7490	39.352	88.702	157.79
0.8	8.9473	38.538	87.935	157.16	9.5683	39.162	88.510	157.59
1.0	8.4516	38,000	87.335	156.42	9.4024	38.987	88.330	157.42
$E_1/E_2 = 15$								
0.0	4.7037	18.810	42.320	75.222	4.7037	18.810	42.320	75.222
0.1	4.6936	18.795	42.295	75.181	4.6987	18.805	42.307	75.202
0.2	4.6707	18.767	42.255	75.131	4.6911	18.792	42.291	75.179
0.3	4.6348	18.721	42.198	75.059	4.6783	18.777	42.271	75.151
0.5	4.5176	18.593	42.049	74.881	4.6400	18.731	42.214	75.082
0.8	4.2483	18.296	41.721	74.509	4.5491	18.629	42.094	74.940
1.0	4.0115	18.030	41.431	74.188	4.4725	18.539	41.992	74.823
$E_1/E_2 = 40$								
0.0	4.0072	16.024	36.040	64.061	4.0072	16.024	36.040	64.061
0.1	3.9938	16.011	36.017	64.021	4.0027	16.015	36.032	64.043
0.2	3.9758	15.980	35.985	63.976	3.9938	16.006	36.013	64.021
0.3	3.9442	15.944	35.933	63.918	3.9848	15.989	35.997	63.994
0.5	3.8432	15.831	35.805	63.757	3.9533	15.953	35.945	63.936
0.8	3.6179	15.576	35.521	63.433	3.8710	15.863	35.845	63.811
1.0	3.4153	15.349	35.271	63.156	3.8105	15.786	35.753	63.708

Table 3. Exact frequency parameters Ω for simply-supported [0, 90] laminated curved thin and moderately thick beams, $a/R = 1.0$, $G_{13}/E_2 = G_{23}/E_2 = 0.5$

	Thin beam theory [eqn (42)]				Moderately thick beam theory [eqns (45)]			
	m				*m*			
a/h	1	2	3	4	1	2	3	4
$E_1/E_2 = 1$ (i.e. single layer)								
0.01	8.4520	38.001	87.335	156.41	8.4508	37.980	87.229	156.07
0.02	8.4520	37.998	87.331	156.41	8.4478	37.919	86.909	155.06
0.05	8.4488	37.985	87.300	156.35	8.4270	37.504	84.790	148.58
0.10	8.4401	37.938	87.183	156.12	8.3546	36.153	78.546	131.57
0.20	8.4047	37.730	86.558	154.40	8.0874	32.122	63.760	98.806
$E_1/E_2 = 15$								
0.01	4.0116	18.031	41.432	74.190	4.0094	18.000	41.286	73.738
0.02	3.9960	17.949	41.227	73.786	3.9885	17.839	40.681	72.095
0.05	3.9471	17.690	40.526	72.278	3.9109	17.089	37.667	64.041
0.10	3.8656	17.212	39.083	68.854	3.7419	15.329	31.300	49.452
0.20	3.7015	16.118	35.403	59.676	3.3312	11.808	21.481	31.295
$E_1/E_2 = 40$								
0.01	3.4145	15.349	35.270	63.157	3.4108	15.298	35.034	62.438
0.02	3.4014	15.274	35.081	62.778	3.3871	15.093	35.220	60.178
0.05	3.3559	15.034	34.424	61.355	3.2932	14.111	30.283	50.017
0.10	3.2809	14.591	33.077	58.157	3.0814	11.964	23.180	35.118
0.20	3.1315	13.592	29.743	49.981	2.5935	3.3979	14.446	20.425

The effect of shear deformation and rotary inertia increases as the orthotropy ratio increases. While the maximum difference (in the fourth frequency) between the frequencies obtained by thin beam equations and moderately thick beam equations is 5.2% for a thickness ratio of 0.05 and single layer beams, this difference is 12.9% for [0, 90] cross-ply beams with an orthotropy ratio of 15 and it is 22.7% for the 40 orthotropy ratio. The difference between both theories becomes large, even for the fundamental frequencies, when the thickness ratio exceeds 0.05, which is taken as the limit of the thin beam theory.

These results can be used in establishing the limits of shell theories and shell equations. They can also be used as benchmarks for future research on using approximate methods, like the Ritz and finite element methods.

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